1

Conservation of Angular momentum:

Isotropic Space → Rotational invariance

Homogeneous space → Linear momentum conserved.
Galilean Invariance

Let us again start with the equation

$$\delta L = \sum_k \frac{\partial L}{\partial q_k}\delta q_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k = 0 \quad \text{— (1)}$$

We know that $\frac{\partial L}{\partial \dot{q}_k} = p_k$ — (2)

Lagrange's equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$ — (3)

From (2) & (3)

$$\frac{d}{dt}(p_k) - \frac{\partial L}{\partial q_k} = 0$$

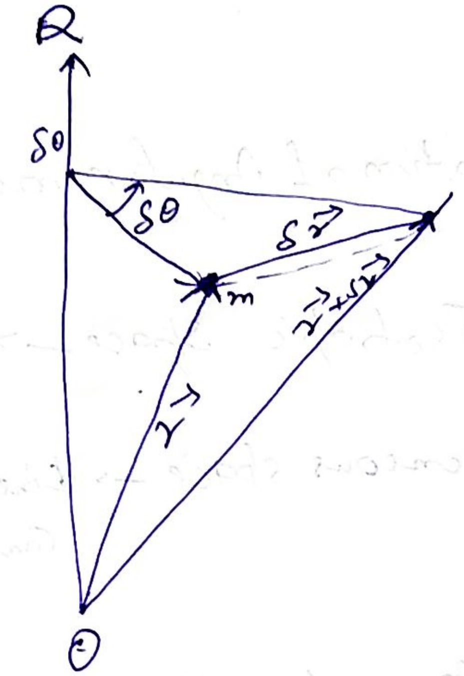
$$\text{or } \dot{p}_k = \frac{\partial L}{\partial q_k} \quad \text{— (4)}$$

Now $\delta L = \sum_k \dot{p}_k \delta q_k + \sum_k p_k \delta \dot{q}_k = 0$ — (5)

Next, we consider a point particle at a distance $\vec{r}$ from the origin O. Let the system is rotated with an angle $\delta\theta$ about an axis. (See fig. on next page)

$m \rightarrow$ mass of the point particle

$\Rightarrow$ Rotation about OQ axis



The change in $\vec{r}$

$$\delta\vec{r} = \delta\vec{\theta}\times\vec{r} \quad -(6)$$

From (6)

change in velocity is given by

$$\delta\dot{\vec{r}} = \delta\vec{\theta}\times\dot{\vec{r}} \quad -(7)$$

Now using Eq. (5), for the present case and writing it in vectorial notation as { Note that here $\vec{p} \equiv p_k$ for $k=1,2,3$; $\dot{\vec{r}} \equiv \dot{q}_k$ }

$$\delta L = \dot{\vec{p}}\cdot\delta\vec{r} + \vec{p}\cdot\delta\dot{\vec{r}} = 0$$

Next, using equation (6) and (7) in above expression, we can write

$$\delta L = \dot{\vec{p}}\cdot(\delta\vec{\theta}\times\vec{r}) + \vec{p}\cdot(\delta\vec{\theta}\times\dot{\vec{r}}) = 0 \quad -(8)$$

Next using the property of Scalar triple product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}),$$ we can

write,

$$\delta\vec{\theta}\cdot(\vec{r}\times\dot{\vec{p}}) + \delta\theta\cdot(\dot{\vec{r}}\times\vec{p}) = 0$$

$$\delta\vec{\theta} \cdot \left[(\vec{r} \times \dot{\vec{p}}) + (\dot{\vec{r}} \times \vec{p}) \right] = 0$$

$$\delta\vec{\theta} \cdot \left[\frac{d}{dt} (\vec{r} \times \vec{p}) \right] = 0$$

Since $\vec{J} = \vec{r} \times \vec{p}$ → angular momentum

$$\therefore \delta\vec{\theta} \cdot \frac{d\vec{J}}{dt} = 0$$

$\delta\theta$ is arbitrary, therefore, we obtain

$$\frac{d\vec{J}}{dt} = 0$$

or $\vec{J} = \text{constant}$

or $\boxed{\vec{J} = \vec{r} \times \vec{p} = \text{constant}}$